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Generalized Mittag–Leffler functions in the theory of finite-size scaling for systems with strong anisotropy and/or long-range interaction

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Abstract

The difficulties arising in the investigation of finite-size scaling in d -dimensional $O(n)$ systems with strong anisotropy and/or long-range interaction, decaying with the interparticle distance r as $r^{-d-\sigma}$ ($0 < \sigma \leq 2$), are discussed. Some integral representations aiming at the simplification of the investigations are presented for the classical and quantum lattice sums that take place in the theory. Special attention is paid to a more general form allowing to treat both cases on an equal footing and in addition cases with strong anisotropic interactions and different geometries. The analysis is simplified further by expressing this general form in terms of a generalization of the Mittag–Leffler special functions. This turned out to be very useful for the extraction of asymptotic finite-size behaviours of the thermodynamic functions.

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1. Introduction

The *standard* finite-size scaling (FSS) theory is usually formulated in terms of *only one* reference length—the bulk correlation length ξ . For a system with finite linear size L , the main statements of the theory are the following.

- (i) The only relevant variable in terms of which the properties of the finite system depend on the neighbourhood of the bulk critical parameter (the temperature in classical systems and the corresponding quantum parameter in quantum systems) driving the phase transition is L/ξ .
- (ii) The rounding of the thermodynamic function exhibiting singularities at the bulk phase transition in a given finite system sets in when $L/\xi = O(1)$.

The tacit assumption is that all other reference lengths are irrelevant and will lead *only to corrections* towards the above picture. Moreover, the crucial point in the finite-size theory is that we always assume that the *finite-size* linear L of systems under consideration and the correlation length ξ are large in the microscopic scale. This means $L \gg a$ and $\xi \gg a$, where a is the lattice spacing. For a comprehensive recent review on this subject and other studies related to it see [1]

To investigate the FSS properties of systems with long-range (LR) interaction decaying at large distances r as $r^{-d-\sigma}$, where d is the space dimensionality and $0 < \sigma \leq 2$ is a parameter controlling the range of the interaction, one needs to extract the finite-size effects from the d -dimensional lattice sum:

$$W_{d,\sigma}^\gamma(t, L) = \sum_{\mathbf{q}} \frac{1}{(t + |\mathbf{q}|^\sigma)^\gamma}, \quad (1.1)$$

where due to the periodic boundary conditions \mathbf{q} is a discrete vector with components $q_i = 2\pi n_i/L$ ($n_i = 0, \pm 1, \pm 2, \dots$), $i = 1, \dots, d$. L is the size of the box confining the system and t is a parameter measuring the distance to the bulk critical point. Here we will not comment on the restrictions on d , σ and γ , nor will we dwell on the problem of convergence of (1.1); these details should be clear from the context. Note that $\sigma = 2$ corresponds formally to the case of short-range (SR) interaction [2]. γ is a parameter allowing to treat classical ($\gamma = 1$) and quantum ($\gamma = 1/2$) systems on an equal footing. Higher values of γ appear in the investigation of finite-size systems to the one loop order in the field theoretical approach. For a recent review on the critical properties of systems with LR interaction see [3, 4].

An other reason for considering $W_{d,\sigma}^\gamma(t, L)$ is, as we will see below, the direct mapping between the lattice sum (1.1) and some lattice sums in combinations with integrals that appear in the theory of systems with strong anisotropic LR interaction of the asymptotic form (see for example [5–9])

$$J(\mathbf{q}) \simeq J(0) + a_{\parallel} |\mathbf{q}_{\parallel}|^\rho + a_{\perp} |\mathbf{q}_{\perp}|^\sigma, \quad (1.2)$$

where the first r directions (called ‘parallel’ and denoted by the subscript \parallel) are extended to infinity and the remaining s directions (called ‘transverse’ and denoted by \perp) are kept finite, with $r + s = d$ and a_{\perp}, a_{\parallel} are metric factors and $\rho, \sigma > 0$. Let us note that there is a limited number of papers that consider FSS assumptions on a microscopic models [5, 6]. It seems that the considerations are mainly on phenomenological level or via computer simulations (see for example [7–9]) because of the problems emerging in analytical treatments.

The study of the difference between the d -dimensional sum (1.1) at large sizes L and its limiting integrals is crucial in the derivation of finite-size effects. In the particular cases $\gamma = 1$ or $1/2$ to solve this problem several approaches have been proposed [5, 10–17]. Among them the most *universal* one is that based on the Poisson summation formula [5, 11, 12, 15, 17, 18]. The aim of this approach¹ is to factorize the d -dimensional sum in the rhs of equation (1.1) and to reduce it to a one-dimensional effective problem. The term $|\mathbf{q}|^\sigma$ in conjunction with γ to be arbitrary in the interval $0 < \gamma < \infty$ causes peculiar mathematical problems concerning the evaluation of the lattice sums over \mathbf{q} . The aim of the present study is to generalize the previous investigations for arbitrary $0 < \gamma < \infty$. By virtue of the relation

$$W_{d,\sigma}^{\gamma+n}(t, L) = \frac{(-1)^n}{\gamma(\gamma+1)\cdots(\gamma+n-1)} \frac{d^n}{d^n t} W_{d,\sigma}^\gamma(t, L), \quad (1.3)$$

one can see that it is formally sufficient to consider only the case $0 < \gamma \leq 1$. Let us first consider separately the classical and the quantum case.

¹ This approach originated from an earlier work of Brézin [19] on FSS in systems with short-range interaction.

1.1. Classical case ($\gamma = 1$)

In the case of classical systems with SR interaction, corresponding to $\gamma = 1$ and $\sigma = 2$, the following substitution is used as an indispensable ingredient for the FSS calculations (see, e.g. [1]):

$$W_{d,2}^1(t, L) = \int_0^\infty dx \exp(-tx) \left[\sum_q \exp(-q^2x) \right]^d, \tag{1.4}$$

where q is the one-dimensional discrete vector. This is the so-called Schwinger parametric representation. The analytic properties of the function $\sum_q \exp(-q^2x)$ are very well known, since it is nothing but the reduced Jacobi θ_3 function. The aim of the above procedure is two-fold: (i) to exponentiate the summand and to reduce the d -dimensional sum to a one-dimensional sum with well-known analytic properties, and (ii) to give the dimensionality d the status of a continuous variable.

In the presence of \mathbf{q}^σ term, it is not so easy to realize this procedure. The problem has been solved by suggesting different generalizations [6, 11, 14, 16, 17] of the Schwinger representation (1.4) that lead to different obstacles.

In order to preserve the possibility for further analytical consideration based on the properties of the reduced Jacobi θ_3 function in [11] the following representation has been used:

$$W_{d,\sigma}^1(t, L) = t^{\frac{2-\sigma}{\sigma}} \int_0^\infty dx Q_\sigma(t^{2/\sigma}x) \left[\sum_q \exp(-q^2x) \right]^d. \tag{1.5}$$

The price one pays for this is that instead of the simple exponent in the integrand of (1.4), the function $Q_\sigma(t)$ appears

$$Q_\sigma(x) = \int_0^\infty dy \exp(-xy) \tilde{Q}_\sigma(y), \tag{1.6a}$$

where

$$\tilde{Q}_\sigma(y) = \frac{1}{\pi} \frac{\sin(\frac{\sigma}{2}\pi) y^{\frac{\sigma}{2}}}{1 + 2y^{\frac{\sigma}{2}} \cos(\frac{\sigma}{2}\pi) + y^\sigma}, \quad 0 < \sigma < 2. \tag{1.6b}$$

First the connection between $Q_\sigma(x)$ and the Mittag–Leffler type functions in the theory of FSS was established in [12]. This reads

$$Q_\sigma(x) = x^{\frac{\sigma}{2}-1} E_{\frac{\sigma}{2}, \frac{\sigma}{2}}^\sigma(-x^{\frac{\sigma}{2}}), \tag{1.7}$$

where $E_{\alpha,\beta}(z)$ are entire functions of the Mittag–Leffler type defined by the power series [20, 21]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad \text{Re}(\alpha) > 0. \tag{1.8}$$

This shows that the study of the finite-size behaviour of the lattice sum

$$W_{d,\sigma}^1(t, L) = \int_0^\infty dx x^{\frac{\sigma}{2}-1} E_{\frac{\sigma}{2}, \frac{\sigma}{2}}^\sigma(-tx^{\frac{\sigma}{2}}) \left[\sum_q \exp(-q^2x) \right]^d \tag{1.9}$$

is a direct consequence of the analytical properties of $E_{\alpha,\beta}(z)$ [12].

1.2. Pure quantum case ($\gamma = 1/2$)

For the investigation of the FSS at zero temperature when the phase transition is driven by a quantum parameter, in (1.1) we have $\gamma = \frac{1}{2}$. Then the following integral representation is obtained:

$$W_{d,\sigma}^{\frac{1}{2}}(t, L) = \frac{2}{\pi} \int_0^\infty dp \sum_{\mathbf{q}} \frac{1}{t + p^2 + |\mathbf{q}|^\sigma}. \quad (1.10)$$

The auxiliary variable p^2 adds an effective extra dimension. Indeed, the pure quantum system corresponds to a $d+1$ -dimensional anisotropic classical system with the geometry of a cylinder $L^d \times \infty$. Recall that the bulk critical behaviour (e.g. critical exponents) of a pure quantum system is equivalent to a $d+z$ - ($z = \frac{\sigma}{2}$ dynamic critical exponent) dimensional classical system (this is the so-called classical to quantum crossover).

On the other hand, in the spirit of relation (1.9) the following modification has been proposed [15]:

$$W_{d,\sigma}^{\frac{1}{2}}(t, L) = \int_0^\infty dx x^{\frac{\sigma}{4}-1} G_{\frac{\sigma}{2}, \frac{\sigma}{4}}(-tx^{\frac{\sigma}{2}}) \left[\sum_{\mathbf{q}} \exp(-q^2 x) \right]^d. \quad (1.11)$$

The new functions $G_{\alpha,\beta}(z)$ are defined by the power series [15]

$$G_{\alpha,\beta}(z) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^\infty \frac{\Gamma(k + \frac{1}{2})}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \alpha, \beta \in \mathbb{C}, \quad \text{Re}(\alpha) > 0. \quad (1.12)$$

Some results on the analytic behaviour of these functions are presented in [5]. In the particular case $\alpha = \frac{\sigma}{2}, \beta = \frac{\sigma}{4}$ the following identity [4]:

$$G_{\frac{\sigma}{2}, \frac{\sigma}{4}}(-z) = \frac{2}{\pi} \int_0^\infty E_{\frac{\sigma}{2}, \frac{\sigma}{2}}(-(z + p^2)) dp, \quad (1.13)$$

can be obtained from equation (1.10) and the relation of its lhs and rhs with the functions $G_{\frac{\sigma}{2}, \frac{\sigma}{4}}(z)$ and $E_{\frac{\sigma}{2}, \frac{\sigma}{2}}(z)$, respectively.

1.3. Anisotropic case ($0 < \gamma < 1$)

In this case, instead of (1.10) we propose the following identity that can be obtained (see appendix A) after some algebra ($0 < \gamma < 1$):

$$W_{d,\sigma}^\gamma(t, L) = \frac{1}{(1-\gamma)\Gamma(\gamma)\Gamma(1-\gamma)} \int_0^\infty dp \sum_{\mathbf{q}} \frac{1}{t + p^{\frac{1}{1-\gamma}} + |\mathbf{q}|^\sigma}. \quad (1.14)$$

Equation (1.14) generalizes the result (1.10) corresponding to the pure quantum case. Here the auxiliary variable $p^{\frac{1}{1-\gamma}}$ acts effectively as an anisotropic extra dimension that generates additional mathematical difficulties. In the denominator of the summand in the rhs of equation (1.14), one can easily recognize the form of the anisotropic interaction (1.2) with $s = d, r = 1$ and $\rho = 1/(1-\gamma)$.

In this paper, we present new representation formulae for the lattice sums defined in equation (1.1) relevant to the investigations of the finite-size scaling properties of a large class of systems: classical, quantum and systems with strong anisotropy. Following the lines of consideration mentioned for the particular cases of the previous subsections 1.1 and 1.2, our aim here is to present functions depending on the three parameters α, β and γ that can play the same role as the functions $E_{\alpha,\beta}(z)$ and $G_{\alpha,\beta}(z)$. The mathematical properties of these functions will be discussed in the next section and some applications will be given.

2. Generalized Mittag–Leffler functions

The following generalization of the Mittag–Leffler functions is defined by the power series [22]:

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \quad \text{Re}(\alpha) > 0, \quad (2.1)$$

are of a significant interest. Here

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma + 1)(\gamma + 2) \cdots (\gamma + k - 1) = \frac{\Gamma(k + \gamma)}{\Gamma(\gamma)}, \quad k = 1, 2, \dots \quad (2.2)$$

These functions are named after Mittag–Leffler who first introduced the particular case with $\beta = \gamma = 1$. Recently, the interest in this type of functions has grown up by their applications in some evolution problems and by their various generalizations appearing in the solution of differential and integral equations. For some mathematical applications see [23, 24] and references therein.

In the present study, we will show that these functions can play an intrinsic role in the theory of FSS. Remark that the generalized functions (2.1) reduce to the $G_{\alpha,\beta}$ given by (1.12) in the particular case $\gamma = \frac{1}{2}$.

One of the most striking properties of these functions is that they obey the following identity:

$$(1 + z)^{-\gamma} = \int_0^\infty dx e^{-x} x^{\beta-1} E_{\alpha,\beta}^\gamma(-x^\alpha z), \quad \text{Re}(\gamma), \text{Re}(\beta) > 0, \quad |z| < 1, \quad (2.3)$$

which is obtained by means of term-by-term integration of series (2.1). As we will show identity (2.3) lies in the basis of the mathematical investigation of FSS in systems with LR interaction. If we set in identity (2.3) $z = y^{-\alpha}$, $y > 0$, and $x = ty$, we will obtain the Laplace transform

$$\frac{y^{\alpha\gamma-\beta}}{(1 + y^\alpha)^\gamma} = \int_0^\infty dt e^{-yt} t^{\beta-1} E_{\alpha,\beta}^\gamma(-t^\alpha) \quad (2.4)$$

from which we derive a new identity by setting $\beta = \alpha\gamma$:

$$\frac{1}{(1 + y^\alpha)^\gamma} = \int_0^\infty dt e^{-yt} t^{\alpha\gamma-1} E_{\alpha,\alpha\gamma}^\gamma(-t^\alpha). \quad (2.5)$$

With the help of the above identity one immediately obtains the relation

$$W_{d,\sigma}^\gamma(t, L) = \int_0^\infty dx x^{\gamma\frac{\sigma}{2}-1} E_{\frac{\sigma}{2},\gamma\frac{\sigma}{2}}^\gamma(-tx^{\frac{\sigma}{2}}) \left[\sum_q \exp(-q^2 x) \right]^d, \quad \text{Re}(\gamma) > 0, \quad (2.6)$$

which is the quested generalization of (1.9) and (1.11). Now it is easy to obtain from equations (1.9), (1.14) and (2.6) the generalization of the integral relation (1.13):

$$E_{\frac{\sigma}{2},\gamma\frac{\sigma}{2}}^\gamma(-z) = \frac{1}{(1 - \gamma)\Gamma(\gamma)\Gamma(1 - \gamma)} \int_0^\infty E_{\frac{\sigma}{2},\frac{\sigma}{2}}^1\left(-\left(z + p^{\frac{1}{1-\gamma}}\right)\right) dp, \quad 0 < \gamma < 1. \quad (2.7)$$

Note that $E_{\frac{\sigma}{2},\frac{\sigma}{2}}^1 := E_{\frac{\sigma}{2},\frac{\sigma}{2}}^\gamma$. The asymptotic expansion for $z \gg 1$ of the generalized Mittag–Leffler functions $E_{\alpha,\beta}^\gamma(z)$ can be obtained from (see appendix B)

$$E_{\alpha,\beta}^\gamma(-z) = \sum_{k=0}^{\infty} (-1)^k \frac{(\gamma)_k}{\Gamma[\beta - \alpha(k + \gamma)]} \frac{z^{-(k+\gamma)}}{k!}, \quad |z| > 1. \quad (2.8)$$

In the particular case $\beta = \alpha\gamma$, relevant to the physical situations we are discussing here, equation (2.8) reduces to

$$E_{\alpha,\alpha\gamma}^\gamma(-x) \simeq -\gamma \frac{x^{-(1+\gamma)}}{\Gamma(-\alpha)}, \quad x \gg 1, \tag{2.9}$$

where only the leading term is accounted for.

In the following section, we will discuss some applications and relations to previous results obtained in the framework of the FSS investigations in the classical and the quantum cases.

3. Finite-size computations

For FSS computations of $O(n)$ systems one needs the large- L behaviour of normalized lattice sums defined by

$$\overline{W}_{d,\sigma}^\gamma(t, L) = \frac{1}{L^d} \sum_{\mathbf{q} \neq \mathbf{0}} \frac{1}{(t + |\mathbf{q}|^\sigma)^\gamma}, \tag{3.1}$$

where γ is relevant to different physical cases. Indeed for integer γ we have classical systems, while for half-integers we have the quantum situation and for $\gamma < 1$ systems with geometry $L^r \times \infty^s$ and strong anisotropy of type (1.2). The last follows directly from the identity (compare with (1.14)):

$$\int_0^\infty dp \sum_{\mathbf{q}} \frac{1}{t + p^\rho + |\mathbf{q}|^\sigma} = \frac{\Gamma(1 - \frac{1}{\rho})\Gamma(\frac{1}{\rho})}{\rho} W_{d,\sigma}^{1-\frac{1}{\rho}}(t, L), \quad \rho > 1. \tag{3.2}$$

The method we use here to extract the large- L behaviour of (3.1) is based upon identity (2.6). After rearrangement of (3.1) we obtain

$$\overline{W}_{d,\sigma}^\gamma(t, L) = \frac{L^{\gamma\sigma-d}}{(2\pi)^{\gamma\sigma}} \int_0^\infty dx x^{\gamma\frac{\sigma}{2}-1} E_{\frac{\sigma}{2},\gamma\frac{\sigma}{2}}^\gamma\left(-\frac{tL^\sigma}{(2\pi)^\sigma} x^{\frac{\sigma}{2}}\right) [A^d(x) - 1], \tag{3.3a}$$

where

$$A(x) \equiv \sum_{n=-\infty}^{+\infty} e^{-xn^2}. \tag{3.3b}$$

For large x , $A(x) - 1$ decreases exponentially and the integral on the right-hand side of equation (3.3a) converges at infinity. For $x \rightarrow 0$, the Poisson transformation formula

$$A(x) = \sqrt{\frac{\pi}{x}} A\left(\frac{\pi^2}{x}\right) \tag{3.4}$$

shows that $A(x)$ converges.

For small x , the integral on the right-hand side of equation (3.3a) has an ultraviolet divergence for $\text{Re}(d) > \gamma\sigma$. So, an analytic continuation in d is required to give a meaning to the integral. Adding and subtracting the small behaviour of the function $A(x)$, we get after straightforward algebra,

$$\overline{W}_{d,\sigma}^\gamma(t, L) = L^{-d+\gamma\sigma} [D_{d,\sigma}^\gamma(tL^\sigma)^{\frac{d-\gamma\sigma}{\sigma}} + F_{d,\sigma}^\gamma(tL^\sigma)], \tag{3.5a}$$

where the constant

$$D_{d,\sigma}^\gamma = \frac{2}{\sigma} \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\gamma - \frac{d}{\sigma}) \Gamma(\frac{d}{\sigma})}{\Gamma(\gamma)\Gamma(\frac{d}{2})} \tag{3.5b}$$

and the functions

$$F_{d,\sigma}^\gamma(y) = \frac{1}{(2\pi)^{\gamma\sigma}} \int_0^\infty dx x^{\gamma\frac{\sigma}{2}-1} E_{\frac{\sigma}{2},\gamma\frac{\sigma}{2}}^\gamma\left(-\frac{y}{(2\pi)^\sigma} x^{\frac{\sigma}{2}}\right) \left[A^d(x) - 1 - \left(\frac{\pi}{x}\right)^{\frac{d}{2}}\right]. \quad (3.5c)$$

The first term in (3.5a) is the bulk contribution (it is L -independent) and the second term is the corresponding finite-size correction. The form (3.5a) is suitable for the investigation of FSS in the vicinity of the critical point i.e. $t \simeq 0$. The function $F_{d,\sigma}^\gamma(y)$ enters in the expressions for the scaling functions of various thermodynamic observables. The dependence on the linear size L of the system is included in the scaling variable y . The behaviour of any thermodynamic function is tightly related to the asymptotic behaviour of $F_{d,\sigma}^\gamma(y)$, which in turn depend upon that of the Mittag–Leffler functions. For detailed discussions of different models with the particular values $\gamma = 1$ and $\gamma = \frac{1}{2}$, the reader is invited to consult references [1, 3, 5, 15, 18], where the finite-size scaling predictions are investigated in great detail. Let us note that at this level, the anisotropy of the scaling behaviour in equation (3.5a) only appears through the parameter γ .

By setting $t = 0$ in (3.5a), we obtain an expression for the finite-size shift of the bulk critical parameter driving the phase transition. This is proportional to $F_{d,\sigma}^\gamma(0)L^{-\lambda}$, where $\lambda = d - \gamma\sigma$ is the shift critical exponent for the specific value of γ . The coefficient $F_{d,\sigma}^\gamma(0)$ can be evaluated analytically as well as numerically for different values of the free parameters d, σ and γ using the method developed in [25].

According to the standard FSS, we *must* have $\lambda = 1/\nu$, where $1/\nu$ is the critical exponent measuring the divergence of the correlation length. The value of ν depends on the concrete microscopic model. For illustration, in the particular case of the symmetric $O(n)$ model in the limit $n \rightarrow \infty$, one can consider two cases: classical and quantum, where $\nu = 1/(d - \sigma)$ and $\nu = 1/(d - \sigma/2)$, respectively. In both cases our result confirms the FSS theory predictions (see, e.g. [1]). Furthermore in order to make contact with the case of strong anisotropy of [6, 26] the effective dimensionality $D = 2d/\sigma + 2/\rho$ must be introduced. It determines the conditions for the phase transition to take place, if the anisotropic LR has the asymptotic form (1.2) with $s = 1, r = d$. In this case $\nu = 2/(\sigma(D - 2))$ and again we have agreement with the FSS theory, provided $2 < D < 4$. Moreover, introducing the effective dimension $D(\gamma) = 2d/\sigma + 2(1 - \gamma)$ we can consider a more general classical system that includes $\gamma = 1, 1/2, (1 - 1/\rho)$ as particular cases.

The finite-size correction to the bulk critical behaviour (of e.g. susceptibility) can be extracted from the asymptotic form of the functions $F_{d,\sigma}^\gamma(y)$ defined by (3.5c) at large argument. This in turn can be obtained with the help of expansion (2.9). After some algebra we get (see appendix C)

$$F_{d,2}^\gamma(y) \simeq -y^{-\gamma} + \left[\frac{d}{2^\gamma (2\pi)^{\frac{d-1}{2}} \Gamma(\gamma)}\right] y^{\frac{1}{4}(d-2\gamma-1)} e^{-\sqrt{y}} \quad (3.6a)$$

for $\sigma = 2$, i.e. for the SR case and

$$F_{d,\sigma}^\gamma(y) \simeq -y^{-\gamma} + \left[2^\sigma \gamma \pi^{-\frac{d}{2}} \frac{\Gamma(\frac{d+\sigma}{2})}{\Gamma(-\frac{\sigma}{2})} \sum_{l \neq 0} \frac{1}{|l|^{d+\sigma}}\right] y^{-(1+\gamma)} \quad (3.6b)$$

for $0 < \sigma < 2$, corresponding to the LR case. Equations (3.6) generalize equations (3.29) of [18] obtained for the particular case $\gamma = 1$.

Equations (3.6) reflect the fact that for systems with LR interaction the finite-size corrections fall-off in *power law* rather than exponential as it is the case for their counterparts with SR interaction. These results are generalizations of those obtained previously in the case of classical systems [12, 13, 17, 18] and those obtained for their quantum counterparts [5, 15].

The former cases can be obtained by using integer values for γ and the latter ones by using half-integer values.

4. Conclusion

We presented some mathematical results on the investigation of the FSS in $O(n)$ systems based on the generalized Mittag–Leffler functions (2.1) that have well-known analytic properties. Mainly two types of systems are of particular interest.

- (i) The fully finite d -dimensional systems with LR interaction decaying algebraically with the interparticle distance.

This is the case with $0 < \sigma < 2$ in (1.1). A special emphasis on the mathematical difficulties arising in the investigation of the FSS both in the classical (with $\gamma = 1$ in (1.1)) and quantum cases (with $\gamma = 1/2$ in (1.1)) is discussed. The used techniques allow the investigations to be simplified and express the results for various thermodynamic quantities in terms of simple, with well-defined analytic properties, mathematical functions. An integral representation (2.6) to deal with such difficulties, at least asymptotically, is presented. It turned out that both cases can be treated on an equal footing.

- (ii) The classical system with mixed geometries with both finite and infinite sizes and strongly anisotropic interaction of type (1.2).

Such type of systems are considered in [6], where $0 < \rho, \sigma < 2$ and $\gamma = 1 - 1/\rho$. An other interesting case is the m -fold Lifshitz point that is characterized by an instability associated with the absence of quadratic terms in the form q_α^2 in the effective Landau–Ginzburg–Wilson Hamiltonian for all $\alpha = 1, 2, \dots, n < d$ [7, 27]. Then in (1.2) one must set $\rho = 4, \sigma = 2$ and $\gamma = 3/4$. This gives a simpler way of solving such problems using generalized Mittag–Leffler functions. It is based on the established mapping (3.2) to a fully finite-size system with specific $0 < \gamma < 1$ in (1.1).

In conclusion, our considerations establish that we can study finite-size scaling behaviour of classical systems, quantum systems and systems with strong anisotropy confined in mixed geometry $\infty \times L^d$, in the framework of a fully finite anisotropic system with a *classical* critical behaviour. This is achieved in a unified fashion, varying the superscript γ in the generalized Mittag–Leffler functions.

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Appendix A. Derivation of equation (1.14)

We have the relation

$$\frac{1}{\mu} p^{-\alpha/\mu} \Gamma\left(\frac{\alpha}{\mu}\right) = \int_0^\infty x^{\alpha-1} e^{-px^\mu} dx, \quad \mu, \operatorname{Re}(\alpha), \operatorname{Re}(p) > 0. \quad (\text{A.1})$$

or if $\alpha/\mu = \gamma$

$$p^{-\gamma} = \frac{\mu}{\Gamma(\gamma)} \int_0^\infty x^{-(1-\gamma\mu)} e^{-px^\mu} dx. \quad (\text{A.2})$$

Using twice (A.2) we get

$$\frac{1}{p^\gamma} = \frac{\mu}{\Gamma(\gamma)} \frac{v}{\Gamma(1-\gamma\mu)} \int_0^\infty dx \int_0^\infty dt e^{-px^\mu} e^{-xt^v} t^{(1-\gamma\mu)v-1}. \quad (\text{A.3})$$

Now on the free parameters μ and ν , we impose the conditions

$$(1 - \gamma\mu)\nu - 1 = 0, \quad \mu = 1, \tag{A.4}$$

and obtain the identity ($\gamma < 1$)

$$\frac{1}{p^\gamma} = \frac{1}{(1 - \gamma)\Gamma(\gamma)\Gamma(1 - \gamma)} \int_0^\infty dt \frac{1}{p + t^{\frac{1}{1-\gamma}}}. \tag{A.5}$$

Equation (1.14) immediately follows from the above identity.

Appendix B. Derivation of the asymptotic behaviour of the generalized Mittag–Leffler functions (2.8)

An integral representation of the generalized Mittag–Leffler functions $E_{\alpha,\beta}^\gamma(z)$ can be obtained with the aid of the Henkel integral for the inverse gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C e^u u^{-z} dz, \tag{B.1}$$

where the integration contour C is a loop which starts and ends at $x = -\infty$ and encircles the origin in the positive sense: $-\pi \leq \arg z \leq \pi$ on C [28]. This enables to get the result

$$E_{\alpha,\beta}^\gamma(z) = \frac{1}{2i\pi\alpha} \int_C dv \frac{e^{v^{1/\alpha}} v^{\gamma-1+(1-\beta)/\alpha}}{(v - z)^\gamma}. \tag{B.2}$$

In the following we will investigate the asymptotic behaviour of the generalized Mittag–Leffler functions at large argument, following the method used in [5]. This may be performed with the aid of the series [29]

$$(x + z)^{-\gamma} = x^{-\gamma} \sum_{k=0}^\infty (-1)^k \frac{(\gamma)_k}{k!} \left(\frac{z}{x}\right)^k, \quad \left|\frac{z}{x}\right| \leq 1; \quad \frac{z}{x} \neq -1. \tag{B.3}$$

After substitution of the latter equation into the integral representation (B.2) one obtains (2.8).

Appendix C. Large asymptotic behaviour of $F_{d,\sigma}^\gamma(y)$ from (3.5c)

To obtain the large y asymptotic behaviour (3.6) of the functions $F_{d,\sigma}^\gamma(y)$ we rewrite (3.5c), with the help of identity (3.4), in the form

$$F_{d,\sigma}^\gamma(y) = \frac{\pi^{\frac{d}{2}}}{(2\pi)^{\gamma\sigma}} \int_0^\infty dx x^{\gamma\frac{\sigma}{2} - \frac{d}{2} - 1} E_{\frac{\sigma}{2}, \gamma\frac{\sigma}{2}}^\gamma \left(-y \frac{x^{\frac{\sigma}{2}}}{(2\pi)^\sigma}\right) \left[A^d \left(\frac{\pi^2}{x}\right) - 1\right] - \frac{1}{(2\pi)^{\gamma\sigma}} \int_0^\infty dx x^{\gamma\frac{\sigma}{2} - 1} E_{\frac{\sigma}{2}, \gamma\frac{\sigma}{2}}^\gamma \left(-y \frac{x^{\frac{\sigma}{2}}}{(2\pi)^\sigma}\right). \tag{C.1}$$

Using the identity

$$\int_0^\infty dx x^{\gamma\frac{\sigma}{2} - 1} E_{\frac{\sigma}{2}, \gamma\frac{\sigma}{2}}^\gamma \left(-x^{\frac{\sigma}{2}}\right) = 1, \quad \sigma > 0, \tag{C.2}$$

from the second term of equation (C.1) we obtain the first terms of equations (3.6).

Further, taking into account the asymptotic behaviour (2.8) of the functions $E_{\alpha,\beta}^\gamma(z)$ and after subsequent integration in the first term of equation (C.1), we obtain finally the asymptotic behaviour given by equations (3.6).

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